

LECTURE 25 BEGINS:

1 Finite Difference

Let $f(x)$ be a given function in x and $y_0, y_1, \dots, y_n$ be its corresponding values at $x_0, x_1, \dots, x_n$, respectively. The x -values are called ‘Arguments’, and y -values are called ‘Entries’. In general, the difference between any two consecutive arguments may not be the same. We can write the given data in the below tabular form for the computational convenience.

Table 1.1					
x	x_0	x_1	x_2	$\dots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\dots$	$f(x_n)$

Then, the values $y_1 - y_0 = f(x_1) - f(x_0)$, $y_2 - y_1$, $y_3 - y_2, \dots$ are called the first order difference and it is denoted by Δy .

Forward difference operator. The operator Δ is called the Forward Difference Operator, where $\Delta y_n = y_{n+1} - y_n$, for examples, $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1, \dots$. Also, $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = y_2 - 2y_1 + y_0$ and higher powers of the operator, when required, can be obtained using the same procedure.

Backward difference operator. The operator ∇ is called the Backward Difference Operator, where $\nabla y_n = y_n - y_{n-1}$, for examples, $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1, \dots$. Also, $\nabla^2 y_2 = \nabla(\nabla y_1) = \nabla(y_1 - y_0) = y_2 - 2y_1 + y_0$ and higher powers of the operator can be obtained following the same steps, when required. In general, $\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$.

For the given data in table 1.1, the k -th term of a sequence is given by

$$y_k = (1 + \Delta)^k y_0 = \left[1 + {}^k C_1 \Delta + {}^k C_2 \Delta^2 + \dots \right] y_0 = y_0 + {}^k C_1 \Delta y_0 + {}^k C_2 \Delta^2 y_0 + \dots \quad (1)$$

Example 1. For the given data, find the values of $\Delta^4 y_0$ and $\Delta^5 y_0$, by constructing a Forward Difference Table.

Table 1.2						
x	1	2	3	4	5	6
$f(x)$	4	15	40	85	156	259

Solution. Using the table 1.3 and the formula we can easily see that $\Delta^4 y_0 = 0$ and $\Delta^5 y_0 = 0$.

Table 1.3						
Forward Difference Table						
x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
1	4	11				
2	15	25	14	6		
3	40	45	20	6	0	
4	85	71	26	6	0	0
5	156	103	32			
6	259					

Example 2. Find the 7th term of the sequence 2, 9, 28, 65, 126, 217 for the nodal points 0, 1, 2, 3, 4, 5, respectively.

Solution. Using the formula (1) and table 1.4, we get

$$\begin{aligned}
 y_6 &= (1 + \Delta)^6 y_0 \\
 &= 1 + {}^6C_1 \Delta y_0 + {}^6C_2 \Delta^2 y_0 + {}^6C_3 \Delta^3 y_0 + {}^6C_4 \Delta^4 y_0 + {}^6C_5 \Delta^5 y_0 + {}^6C_6 \Delta^6 y_0 \\
 &= 2 + 6 \cdot 7 + 15 \cdot 12 + 20 \cdot 6 + 15 \cdot 0 + 6 \cdot 0 + 1 \cdot 0 = 344.
 \end{aligned}$$

Table 1.4							
Forward Difference Table							
x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	2	7					
1	9	19	12	6			
2	28	37	18	6	0		
3	65	61	24	6	0	0	0
4	126	91	30				
5	217						

2 Interpolation

Let $x_0, x_1, \dots, x_n$ be the nodal points such that $x_1 - x_0 = x_2 - x_1 = \dots = x_n - x_{n-1}$, i.e., the points are equidistant. Let $y = f(x)$ be a function, whose values are $y_0, y_1, \dots, y_n$ at $x_0, x_1, \dots, x_n$, respectively.

Our goal is to find a function $f(x)$ such that $y_i = f(x_i)$ must be satisfied. From the sequence of functions $\{f(x)\}$, there is unique n th degree polynomial $P_n(x)$ such that $y_i = P_n(x_i), \forall i = 0, 1, \dots, n$. This $P_n(x)$ is called ‘Interpolating Polynomial’ such that $|f(x) - P_n(x)| < \varepsilon$.

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2.1 Newton’s Forward Interpolation

Let $x_0, x_1, \dots, x_n$ be equidistant and let $y_0 = f(x_0), y_1 = f(x_1), \dots, y_n = f(x_n)$. Also, suppose that $x_1 - x_0 = h, x_2 - x_1 = h, \dots, x_n - x_{n-1} = h$. Let $P_n(x)$ be a polynomial of degree n such that $y_i = f(x_i) = P_n(x_i), \forall i = 0, 1, \dots, n$. Further, let us assume that

$$P_n(x) = a_0 + a_1(x - x_0)^{(1)} + \dots + a_r(x - x_0)^{(r)} + \dots + a_n(x - x_0)^{(n)}, \quad (2)$$

where $x^{(n)}$ is called the factorial polynomial and it is defined by

$$x^{(n)} = x(x - h)(x - 2h) \dots (x - \overline{n-1}h).$$

In particular, $(x - x_0)^{(1)} = x - x_0$, $(x - x_0)^{(2)} = (x - x_0)(x - \overline{x_0 + h})$, $(x - x_0)^{(3)} = (x - x_0)(x - \overline{x_0 + h})(x - \overline{x_0 + 2h})$, etc.

The constants in (2) can be determined by

$$\begin{aligned} a_0 &= P_n(x), \\ a_1 &= \frac{P_n(x_1) - P_n(x_0)}{(x_1 - x_0)^{(1)}} + \dots + a_n \frac{(x_1 - x_0)^{(n)}}{(x_1 - x_0)^{(1)}}, \\ a_r &= \frac{\Delta^r P_n(x)}{r!h^r} + \left[\text{term containing } \frac{(x_r - x_0) \text{ as the factors}}{r!h^r} \right]. \end{aligned} \quad (3)$$

Next, we set $x = x_0$ in the general formula (3) and get

$$a_r = \frac{\Delta^r P_n(x_0)}{r!h^r} = \frac{\Delta^r y_0}{r!h^r}. \quad (4)$$

Using (2) and (4), the required polynomial has the form:

$$P_n(x) = y_0 + \frac{(x - x_0)}{1!h} \Delta y_0 + \frac{(x - x_0)(x - \overline{x_0 + h})}{2!h^2} \Delta^2 y_0 + \dots$$

Substituting u at the place of $\frac{(x - x_0)}{h}$, i.e., $x = x_0 + uh$ in the above formula to get

$$P_n(x) = P_n(x_0 + uh) = y_0 + \frac{u}{1!} \Delta y_0 + \dots + \frac{u(u-1) \dots (u-r+1)}{r!} \Delta^r y_0 + \dots \quad (5)$$

Equation (5) is called the Newton’s forward interpolation formula.

Example 1. Find the value of y at $x = 21$ from the below table.

Table 2.1				
Forward Difference Table				
x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.3420	0.0487		
23	0.3907	0.0477	-0.0010	
26	0.4384	0.0464	-0.0013	-0.0003
29	0.4845			

Solution. Using $u = \frac{x-x_0}{h} = \frac{21-20}{3} = \frac{1}{3}$ and the formula (5), we get

$$\begin{aligned}
y = P_3(x) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \\
&= 0.3420 + \frac{1}{3} \cdot 0.0487 + \frac{1}{3} \frac{(-2)}{3} \frac{1}{2!} \cdot (-0.0010) + \frac{1}{3} \frac{(-2)}{3} \frac{(-5)}{3} \frac{1}{3!} \cdot (-0.0003) \\
&= 0.3583.
\end{aligned}$$

Example 2. Find the value of $\sin 52^\circ$ using the table 2.2.

Table 2.2				
Forward Difference Table				
x	$f(x) = \sin x$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
45°	0.7071	0.0589		
50°	0.7660	0.0532	-0.0057	
55°	0.8192	0.0468	-0.0064	-0.0007
60°	0.8660			

Solution. Using the below formula

$$y = P_3(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

and $u = \frac{x-x_0}{h} = \frac{52^\circ-45^\circ}{5^\circ} = 1.4^\circ$, complete the solution.

2.2 Newton's Backward Interpolation

Following the same process as in previous subsection, the Newton's Backward Interpolation Formula has the below form, i.e., the interpolating polynomial is give by

$$P_n(x) = P_n(x_0 + vh) = y_0 + \frac{v}{1!}\nabla y_n + \dots + \frac{v(v+1)\dots(v+r-1)}{r!}\Delta^r y_0 + \dots \quad (6)$$

where $v = \frac{x-x_0}{h}$

Example 1. The population in a town is given below:

Table 2.3					
Forward Difference Table					
x	$f(x) = \sin x$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
1941	20	4			
1951	24	5	1		
1961	29	7	2	1	
1971	36	10	3	1	0
1981	46				

Then, find the population of town in 1946 and 1976.

Solution. Part I: We know $u = \frac{x-x_0}{h} = \frac{1946-1941}{10} = \frac{1}{2}$, and the formula

$$y = P_n(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$\Rightarrow y(x = 1946) = P_n(1946) = ?$$

Part II: We know $v = \frac{x-x_0}{h} = \frac{1976-1981}{10} = -\frac{1}{2}$ and therefore using table 2.4, we get

Table 2.4					
Forward Difference Table					
x	$f(x) = \sin x$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1941	20	4			
1951	24	5	1		
1961	29	7	2	1	
1971	36	10	3	1	0
1981	46				

$$y = P_n(x) = y_4 + v\nabla y_4 + \frac{v(v+1)}{2!}\nabla^2 y_4 + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_4 + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_4$$

$$\Rightarrow y(x = 1976) = P_n(1976) = 40.5622.$$

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2.3 Lagrange Interpolation

Let us consider the data table 1.1. Unlike the above cases, here we have, $x_1 - x_0 \neq x_2 - x_1 \neq x_3 - x_2 \neq \dots$ and so on.

The interpolating polynomial is

$$\begin{aligned} f(x) &= P_n(x) \\ &= a_0(x - x_1)(x - x_2) \dots (x - x_n) + a_1(x - x_0)(x - x_2) \dots (x - x_n) \\ &\quad + \dots + a_n(x - x_0)(x - x_2) \dots (x - x_{n-1}) \end{aligned}$$

where,

$$\begin{aligned} a_0 &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}, \\ &\vdots \\ a_n &= \frac{f(x_n)}{(x_n - x_0)(x_n - x_2) \dots (x_n - x_{n-1})}. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= P_n(x) \\ &= \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} f(x_1) \\ &\quad + \dots + \frac{(x - x_0)(x - x_2) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_2) \dots (x_n - x_{n-1})} f(x_n). \end{aligned} \quad (7)$$

Example 1. Given the below data, using Lagrange Interpolation formula, find the value of $y(10)$.

Table 2.5				
x	5	6	9	11
$f(x)$	12	13	14	16

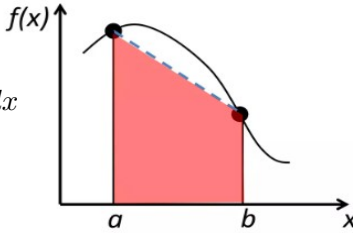
Solution. We have the required interpolating polynomial as follows:

$$\begin{aligned} y &= P_n(x) = P_3(x) \\ &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) \\ &\quad + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3) \\ \Rightarrow y(10) &= \frac{(10 - 6)(10 - 9)(10 - 11)}{(5 - 6)(5 - 9)(5 - 11)} 12 + \frac{(10 - 5)(10 - 9)(10 - 11)}{(6 - 5)(6 - 9)(6 - 11)} 13 \\ &\quad + \frac{(10 - 5)(10 - 6)(10 - 11)}{(9 - 5)(9 - 6)(9 - 11)} 14 + \frac{(10 - 5)(10 - 6)(10 - 9)}{(11 - 5)(11 - 6)(11 - 9)} 16 \\ &= 14.6666. \end{aligned}$$

3 Numerical Integration

3.1 Trapezoidal Rule

Let $x_0, x_1, \dots, x_n$ are equidistant nodal points. Also, say, the step length is $h = x_n - x_{n-1}$, then

$$\begin{aligned}
 I &= \int_a^b f(x) dx = \int_{x_0}^{x_n=x_0+nh} f(x) dx \\
 &= \int_{x_0}^{x_0+h} f(x) dx + \int_{x_0+h}^{x_0+2h} f(x) dx + \dots + \int_{x_0+(n-1)h}^{x_0+nh} f(x) dx \\
 &= \frac{y_0 + y_1}{2}h + \frac{y_1 + y_2}{2}h + \dots + \frac{y_n + y_{n-1}}{2}h \\
 &= \frac{y_0 + y_n}{2}h + h(y_1 + y_2 + \dots + y_{n-1}) \\
 &= \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]. \quad (8)
 \end{aligned}$$


The equation (8) is called the Trapezoidal Rule.

Example 1. Find the value of $\int_{-3}^3 x^4 dx$ by Trapezoidal rule.

Solution. Here, $y = f(x) = x^4$. The given interval is $[-3, 3]$. Then, $h = \frac{3-(-3)}{6} = 1$ as we have considered $n = 6$ sub-intervals.

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

$$\begin{aligned}
 I &= \int_{-3}^3 x^4 dx \\
 &= \frac{1}{2} [(81 + 81) + 2(16 + 1 + 0 + 1 + 16)] \\
 &= \frac{1}{2} [162 + 68] \\
 &= 115.
 \end{aligned}$$

The exact value of the integration: $I = \int_{-3}^3 x^4 dx = \left[\frac{x^5}{5} \right]_{-3}^3 = 97.2$.

3.2 Simpson's $\frac{1}{3}$ rd Rule

Lastly, we briefly give the formula for the Simpson's $\frac{1}{3}$ rd Rule as follows:

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} [(y_0 + y_n) + 2(y_2 + y_4 + \dots) + 4(y_1 + y_3 + \dots)]. \quad (9)$$

Example 1. Proceeding with same example and using Simpson's $\frac{1}{3}$ rd rule.

Solution. In short, we get

$$\begin{aligned} I &= \int_{-3}^3 x^4 dx \\ &= \frac{h}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3)] \\ &= \frac{1}{3} [(81 + 81) + 2(1 + 1) + 4(16 + 0 + 16)] \\ &= \frac{h}{3} [162 + 4 + 128] \\ &= 98. \end{aligned}$$

Which is clearly more closer to the exact value of the integration than the Trapezoidal rule's outcome.

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